

**18** *Years*

*Previous Years Solved Papers*

# **Civil Services Main Examination**

(2009-2026)

## **Mathematics**

Paper-II

*Topicwise Presentation*



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**Civil Services Main Examination Previous Solved Papers : Mathematics Paper-II**

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# Preface

**Civil Service** is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.



**B. Singh** (Ex. IES)

I feel extremely glad to launch the fourth edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes

**B. Singh (Ex. IES)**

CMD, MADE EASY Group



Previous Years Solved Papers of  
**Civil Services Main Examination**

**Mathematics : Paper-II**

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## 1. Groups

- 1.1 If  $\mathbb{R}$  is the set of real number and  $\mathbb{R}_+$  is the set of positive real numbers, show that  $\mathbb{R}$  under addition  $(\mathbb{R}, +)$  and  $\mathbb{R}_+$  under multiplication  $(\mathbb{R}_+, \cdot)$  are isomorphic. Similarly, if  $\mathbb{Q}$  is the set of rational numbers and  $\mathbb{Q}_+$  the set of positive rational numbers, are  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}_+, \cdot)$  isomorphic? Justify your answer.

(2009 : 4+8 =12 Marks)

Solution:

Let  $\mathbb{R}$  be the set of real number and  $\mathbb{R}_+$  be the set of positive real number.

We have to show

$$(\mathbb{R}, +) \cong (\mathbb{R}_+, \cdot)$$

Define  $f: \mathbb{R} \rightarrow \mathbb{R}_+$  as

$$f(x) = a^x; \text{ where } a > 0.$$

We will show  $f$  is one-one.

Consider,

$$\begin{aligned} \ker f &= \{x \in \mathbb{R} \mid f(x) = 1\} \\ &= \{x \in \mathbb{R} \mid a^x = 1\} \\ &= \{x \in \mathbb{R} \mid x = \log_a 1\} \\ &= \{x \in \mathbb{R} \mid x = 0\} \\ &= \{0\} \end{aligned}$$

$\therefore f$  is 1-1.

We will show  $f$  is homomorphism.

Let  $x, y \in \mathbb{R}$

Consider

$$\begin{aligned} f(x+y) &= a^{x+y} \\ &= a^x \cdot a^y = f(x) \cdot f(y) \end{aligned}$$

$\therefore f$  is homomorphism. We will show  $f$  is onto, i.e., we have to find for any positive real number 'y' some real number  $x$  such that

$$f(x) = y$$

i.e.,

$$a^x = y$$

As

$$a^x = y$$

On taking log both sides

$\Rightarrow$

$$x = \log_a y$$

$\therefore$

$$f(x) = y$$

Hence,  $f$  is onto.

$\therefore$

$$(\mathbb{R}, +) \cong (\mathbb{R}_+, \cdot)$$

Let  $\mathbb{Q}$  be the set of rational numbers and  $\mathbb{Q}_+$  be the set of positive rational number.

If  $f$  is homomorphism from  $\mathbb{Q}$  to  $\mathbb{Q}_+$ , then

$$f(x, y) = f(x)f(y) \quad \forall x, y \in Q$$

And if image of 1 is known then the image of every element will be known.

$$\therefore f(x) = a^x \text{ where } a = f(1)$$

$$\text{If } a = 1, \quad f(x) = 1$$

$\therefore f$  is trivial homomorphism.

$$\text{If } a \neq 1, \quad f(x) = 1$$

$$\text{then } f(x) = a^x \in Q_+ \quad \forall x \in Q$$

which is a contradiction.

Hence, only trivial homomorphism is possible.

$$\therefore (Q, +) \not\cong (Q_+, \cdot)$$

- 1.2 Determine the number of homomorphisms from the additive group  $Z_{15}$  to the additive group  $Z_{10}$ . ( $Z_n$  is the cyclic group of order  $n$ ).**

(2009 : 12 Marks)

**Solution:**

Let  $\phi : Z_{15} \rightarrow Z_{10}$  be a homomorphism.

As  $Z_{15}$  is a cyclic group of order 15.

$$Z_{15} = \langle 1 \rangle$$

Under homomorphism, if element 1 will be mapped then remaining elements will get mapped themselves ( $\because G$  is cyclic)

$$\text{Suppose, } \phi(1) = x$$

As we know, if  $f$  is homomorphism from  $G$  to  $G'$  then  $O(f(a)) \mid O(a)$  where  $a \in G$ .

$$\text{As } \phi(1) = x.$$

$$\therefore O(x) \mid O(1) = 15$$

And order of element divides order of group

$$\therefore O(x) \mid 10 \text{ As } O(x) \mid 15$$

$$O(x) \mid 15 \Rightarrow O(x) \mid \text{g.c.d}(15, 10) = 5$$

$$\therefore O(x) = 1 \text{ or } O(x) = 5$$

If  $O(x) = 1$ . Then it is trivial homomorphism. And if  $O(x) = 5$ .

**Note :** In  $Z_n$ , number of elements of order  $k = \phi(k)$ ; provided  $k \mid n$ .

$$\therefore \text{In } Z_{10}, \text{ number of elements of order } 5 = \phi(5) = 4.$$

$\therefore$  We have 4 possibilities for  $x$ .

$$\text{Total number of homomorphism} = 4 + 1 = 5.$$

- 1.3 Show that the alternating group on four letters  $A_4$  has no subgroup of order 6.**

(2009 : 15 Marks)

**Solution:**

Consider the alternating group  $A_4$ .

$$\sigma(A_4) = \frac{\sigma(S_4)}{2} = \frac{12}{2} = 6$$

We show although  $6 \mid 12$ ,  $A_4$  has no subgroup of order 6. Suppose  $H$  is a subgroup of  $A_4$  and  $\sigma(H) = 6$ .

By previous problem the number of distinct 3-cycles in  $S_4$  is

$$\frac{1}{3} \cdot \frac{4!}{(4-3)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 1} = 8$$

Again, as each 3-cycle will be even permutation all these 3-cycles are in  $A_4$ . Obviously then, at least one 3-cycle, say  $s$  does not belong to  $H$  ( $\sigma(H) = 6$ ).

Now,  $\sigma \in H \Rightarrow \sigma^2 \in H$ , because if  $\sigma^2 \in H$ .

Then,  $\sigma^4 \in H$

$\Rightarrow \sigma \in H$

As  $\sigma^3 = I$  as  $\sigma(\sigma) = 3$

Let  $K = \langle \sigma \rangle = \{I, \sigma, \sigma^2\}$  then,  $\sigma K = 3 (= \sigma(\sigma))$

and  $H \cap K = \{1\}$  ( $\sigma, \sigma^2 \notin H$ )

$$\Rightarrow \sigma(HK) = \frac{\sigma(H) \cdot \sigma(K)}{\sigma(H \cap K)} = \frac{6 \cdot 3}{1} = 18,$$

not possible as  $HK \subseteq A_4$  and  $\sigma(A_4) = 12$ .

**1.4** Let  $G = \mathbb{R} - \{-1\}$  be the set of all real numbers omitting  $-1$ . Define the binary relation  $*$  on  $G$  by  $a*b = a + b + ab$ . Show  $(G, *)$  is a group and it is abelian.

(2010 : 12 Marks)

**Solution:**

Given : Binary relation  $*$  as

$$a*b = a + b + ab, \text{ where } a, b \in G.$$

**Closure :** Let  $a, b \in G$

$$\therefore a*b = a + b + ab$$

if  $a + b + ab = -1$ , then

$$a + b + ab + 1 = 0$$

$$\Rightarrow (1 + a) + b(1 + a) = 0$$

$$\Rightarrow (1 + a) + (1 + b) = 0$$

$$\Rightarrow \text{either } 1 + a = 0 \Rightarrow a = -1$$

$$\text{or } 1 + b = 0 \Rightarrow b = -1$$

$\therefore$  both  $a, b \in G$

$$\therefore a \neq -1 \text{ and } b \neq -1$$

$$\therefore a + b + ab \neq -1 \text{ for any } a, b \in G$$

$$\therefore a + b + ab \neq -1 \text{ for any } a, b \in G$$

$$\therefore a*b \in G$$

So, closure is satisfied.

**Associative :** Let  $a, b, c \in G$

$$\begin{aligned} \therefore (a*b)*c &= (a + b + ab)*c \\ &= a + b + ab + c + (a + b + ab)c \\ &= a + b + ab + c + ac + bc + abc \\ &= a + b + c + ab + ac + bc + abc \end{aligned}$$

$$\begin{aligned} \text{Also, } a*(b*c) &= a*(b + c + bc) \\ &= a + b + c + bc + a(b + c + bc) \\ &= a + b + c + ab + bc + ac + abc \end{aligned}$$

$$\text{as } (a*b)*c = a*(b*c)$$

$\therefore$  Associative property is satisfied.

**Identity :**

$$\text{Let } a*b = a = a + b + ab$$

$$\begin{aligned} \Rightarrow a + b + ab &= a \\ \Rightarrow b(1 + a) &= 0 \\ \text{as } a \neq -1 &\Rightarrow b = 0 \end{aligned}$$

$\therefore b = 0$  is an identity and as  $O \in G$ .

$\therefore$  identity exists.

**Inverse :**

Let  $a * b = 0 = a + b + ab$

$$\Rightarrow a(1 + b) = -b \Rightarrow a = \frac{-b}{1+b} \quad (b \neq -1)$$

Also,  $\frac{-b}{1+b} \in G$

$\therefore$  Inverse exists.

As closure, associative property, identity, inverse conditions are satisfied.  $\therefore (G, *)$  is a group.

Now,  $a * b = a + b + ab$   
 $b * a = b + a + ba = a + b + ab$   
 as  $a * b = b * a$

$\therefore (G, *)$  is abelian.

- 1.5 Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this? Justify.

(2010 : 12 Marks)

**Solution:**

Let  $z_6$  is cyclic group of order 6  
 $z_2$  is cyclic group of order 2  
 $z_3$  is cyclic group of order 3

Now, we know that

$$Z_m \times Z_n \cong Z_{mn}$$

when  $m$  and  $n$  are co-prime.

Here,  $m = 2, n = 3$

$$\therefore Z_2 \times Z_3 \cong Z_6$$

- 1.6 Let  $(\mathbb{R}^*, \cdot)$  be the multiplicative group of non-zero real and  $(GL(n, \mathbb{R}), X)$  be the multiplicative group of  $n \times n$  non-singular real matrices. Show that the quotient group  $GL(n, \mathbb{R}) / SL(n, \mathbb{R})$  and  $(\mathbb{R}^*, \cdot)$  are isomorphic where

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) / \det A = 1\}$$

What is the center of  $GL(n, \mathbb{R})$ ?

(2010 : 15 Marks)

**Solution:**

Given,  $(GL(n, \mathbb{R}), X)$  is multiplicative group of real matrices.

Let  $f: GL(n, \mathbb{R}) \rightarrow (\mathbb{R}^*, \cdot)$  be a homomorphism where for any  $A \in G$ .

$$\begin{aligned} \phi(A) &= |A| \\ \therefore \phi(A \times B) &= |AB| = |A| |B| \end{aligned}$$

Now, let  $\text{Ker}(\phi)$  contains matrices such it  $B \in \text{Ker}(\phi)$

then  $\phi(AB) = A$

$$\therefore |B| = 1$$

$\therefore \text{Ker}(\phi)$  contains matrices such that  $|B| = 1$  if  $B \in \text{Ker}(\phi)$ .

$\therefore$  By first fundamental theorem of homomorphism,

$$\frac{GL}{\text{Ker}(\phi)} \cong (\mathbb{R}^*, \cdot)$$

Given,

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) / |A| = 1\}$$

i.e.,  $SL(n, \mathbb{R})$  is  $\text{Ker}(\phi)$ .

$$\therefore \frac{GL(n, \mathbb{R})}{SL(n, \mathbb{R})} \cong (\mathbb{R}^*, \cdot)$$

Now, let  $z$  be the center of  $GL(n, \mathbb{R})$ .

$$\text{for } x \in z \text{ where } X = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$\forall A \in GL(n, \mathbb{R}),$$

$$AX = XA$$

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ where } A \in GL(n, \mathbb{R})$$

Now,

$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

$$XA = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ap+qc & pb+qd \\ ar+cs & br+ds \end{bmatrix}$$

$\therefore$  for  $XA = AX$

$$\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} = \begin{bmatrix} ap+qc & pb+qd \\ ar+cs & br+ds \end{bmatrix}$$

Holds true when  $p = q = r = s$

or

$$p = s, q = r = 0$$

$\therefore$

$$z = \left\{ \begin{bmatrix} p & p \\ p & p \end{bmatrix}, \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \mid p \in R \right\}$$

### 1.7 Show that the set

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

of six transformations on the set of Complex numbers defined by

$$f_1(z) = z, f_2(z) = 1 - z, f_3(z) = \frac{z}{(z-1)},$$

$$f_4(z) = \frac{1}{z}, f_5(z) = \frac{1}{(1-z)} \text{ and } f_6(z) = \frac{(z-1)}{z}$$

is a non-abelian group of order 6 w.r.t. composition of mappings.

(2011 : 12 Marks)

**Solution:**

Let us construct the following table of the elements of  $G$  w.r.t. composition of mappings :

|       | $f_1$           | $f_2$           | $f_3$           | $f_4$           | $f_5$           | $f_6$           |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $f_1$ | $z$             | $1-z$           | $\frac{z}{z-1}$ | $\frac{1}{z}$   | $\frac{1}{1-z}$ | $\frac{z-1}{z}$ |
| $f_2$ | $1-z$           | $z$             | $\frac{1}{1-z}$ | $\frac{z-1}{z}$ | $\frac{z}{z-1}$ | $\frac{1}{z}$   |
| $f_3$ | $\frac{z}{z-1}$ | $\frac{z-1}{z}$ | $z$             | $\frac{1}{1-z}$ | $\frac{1}{z}$   | $1-z$           |
| $f_4$ | $\frac{1}{z}$   | $\frac{1}{1-z}$ | $\frac{z-1}{z}$ | $z$             | $1-z$           | $\frac{z}{z-1}$ |
| $f_5$ | $\frac{1}{1-z}$ | $\frac{1}{z}$   | $1-z$           | $\frac{z}{z-1}$ | $\frac{z-1}{z}$ | $z$             |
| $f_6$ | $\frac{z-1}{z}$ | $\frac{z}{z-1}$ | $\frac{1}{z}$   | $1-z$           | $z$             | $\frac{1}{1-z}$ |

From the table it is clear that

$$(f_i \circ f_i)(z) = (f_i \circ f_i)(z) \quad \forall z \in C, 1 \leq i \leq 6$$

$\therefore f_i$  is the identity of  $G$ .

Also,

$\Rightarrow G$  is a group.

But

$$(f_2 \circ f_3)(z) = \frac{1}{1-z}$$

and

$$(f_3 \circ f_2)(z) = \frac{z-1}{z}$$

$\therefore G$  is a non-abelian group.

1.8 (i) Prove that a group of Prime order is abelian.

(2011 : 6 Marks)

(ii) How many generators are there of the cyclic group  $(G, \cdot)$  of order 8?

(2011 : 6 Marks)

Solution:

(i) Let  $O(G) = P$ , where  $P$  is a prime number.

Let  $a \in G$  be any element.

Since order of an element divides the order of the group.

$$\therefore O(a) \mid O(G) = P$$

$$\Rightarrow O(a) = 1 \text{ or } O(a) = P$$

If  $a \neq e$ , then

$$O(a) = P$$

Let

$$H = \langle a \rangle$$

$\Rightarrow$

$$O(H) = O(a) = P \text{ and } H \subseteq G$$

$\therefore$

$$H = G = \langle a \rangle$$

$\Rightarrow G$  is cyclic.

$\Rightarrow G$  is abelian (as every cyclic group is abelian).

(ii) Let  $G$  be a cyclic group of order 8.

$\therefore \exists$  an element  $a \in G$  such that  $O(a) = 8$

We know that,

$$O(a^n) = \frac{O(a)}{(O(a), n)}$$

$\therefore$

$$O(a^n) = O(a) \Rightarrow (O(a), n) = 1$$

Number of elements of  $G$  whose order is co-prime to

$$\begin{aligned} O(a) &= \phi(O(a)) \\ &= \phi(8) = \phi(2^3) \\ &= 4 \end{aligned} \quad (\because \phi(P^n) = P^n - P^{n-1})$$

$\therefore$  Number of generators of cyclic group of order 8 = 4.

**1.9 Give an example of a group  $G$  in which every proper subgroup is cyclic but the group itself is not cyclic.**

(2011 : 15 Marks)

**Solution:**

Let  $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$   
 Define product on  $G$  by usual multiplication together with  
 $i^2 = j^2 = k^2 = -1$   
 $ij = -ji = k$   
 $jk = -kj = i$   
 $ki = -ik = j$

Then  $G$  forms a group.

But  $G$  is not abelian as  $ij \neq ji$ .

$\Rightarrow G$  can not be cyclic as every cyclic group is abelian.

Let

$$\begin{aligned} H_1 &= [1] = \langle 1 \rangle \\ H_2 &= [\pm 1] = \langle -1 \rangle \\ H_3 &= [\pm 1, \pm i] = \langle i \rangle = \langle -i \rangle \\ H_4 &= [\pm 1, \pm j] = \langle j \rangle = \langle -j \rangle \\ H_5 &= [\pm 1, \pm k] = \langle k \rangle = \langle -k \rangle \end{aligned}$$

Thus,  $H_1$  to  $H_5$  are proper subgroups of  $G$  and all there are cyclic. Hence the result.

**1.10 Let  $a$  and  $b$  be elements of a group, with  $a^2 = e$ ,  $b^6 = e$  and  $ab = b^4a$ . Find the order of  $ab$ , and express its inverse in each of the forms  $a^m b^n$  and  $b^m a^n$ .**

(2011 : 20 Marks)

**Solution:**

Given :

$$a^2 = e, b^6 = e \text{ and } ab = b^4a$$

$\therefore$

$$ab = b^4a$$

$\Rightarrow$

$$aba^{-1} = b^4aa^{-1} = b^4e = b^4$$

Again,

$$b^8 = b^4 \cdot b^4$$

$$= (aba^{-1})(aba^{-1}) = ab^2a^{-1}$$

$\Rightarrow$

$$b^{16} = b^8 \cdot b^8 = (ab^2a^{-1})(ab^2a^{-1})$$

$$= ab^4a^{-1}$$

$$= a(aba^{-1})a^{-1}$$

$$(\because b^4 = aba^{-1})$$

$$= a^2ba^{-2}$$

$$= a^2b(a^2)^{-1}$$

$$= ebe = b$$

$\Rightarrow$

$$b^{16} = b \Rightarrow b^{15} = e$$

$\Rightarrow$

$$O(b) = 1, 3, 5 \text{ or } 15$$

...(i)

Also, given

$$b^6 = e$$

$\Rightarrow$

$$O(b) = 1, 2, 3 \text{ or } 6$$

...(ii)

$\therefore$  From (i) and (ii)

$$O(b) = 3$$

And

$$ab = b^4a = b^3ba = eba = ba$$

Also, we know that, for  $a, b \in G$  such that  $ab = ba$  and  $(O(a), O(b)) = 1$

then

$$O(ab) = O(a) \cdot O(b)$$

$\therefore$

$$O(ab) = 2 \times 3 = 6$$

$\Rightarrow$

$$(ab)^6 = e$$

$\Rightarrow$

$$(ab)^5 = (ab)^{-1}$$

$\Rightarrow$

$$a^5 b^5 = (ab)^{-1}$$

Similarly, as  $ab = ba$ ,

$$(ab)^{-1} = b^5 a^5$$

**1.11 How many elements of order 2 are there in the group of order 16 generated by  $a$  and  $b$  such that the order of  $a$  is 8, the order of  $b$  is 2 and  $bab^{-1} = a^{-1}$ .**

(2012 : 12 Marks)

**Solution:**

Let  $G$  be the given group of order 16.

Then,

$$O(G) = 16$$

$$O(a) = 8$$

$$O(b) = 2$$

and

$$bab^{-1} = a^{-1}$$

...(i)

From (i), we have

$$b(bab^{-1}) = b(a^{-1})$$

$\Rightarrow$

$$b^2 ab^{-1} = ba^{-1}$$

$\Rightarrow$

$$ab^{-1} = ba^{-1}$$

$\Rightarrow$

$$ab = ba^{-1}$$

$$(\because O(b) = 2 \Rightarrow b^2 = e)$$

$$(\because b^2 = e \Rightarrow b = b^{-1})$$

and  $H$  is generated by  $a$  and  $b$ , where  $O(a) = 8$ ,  $O(b) = 2$ ,  $bab^{-1} = a^{-1}$ .

$\therefore G = D_8$ , dihedral group of order 8.

Again, let

$$H = \langle a \rangle$$

Since,

$$O(a) = 8$$

$\therefore H$  is a cyclic group of order 8. Also, in a finite cyclic group, for each divisor  $\frac{k}{n} = O(G)$ , there exist exactly

$\phi(k)$  elements of order  $k$ .

$[\phi(n) = \text{positive integers upto a given integer } n \text{ that are relatively prime to } n].$

As  $2 \mid 8$ ,  $\therefore \phi(2)$  elements of order 2, i.e., 1 element of order 2 ( $\because \phi(2) = 1$ )

Also,  $D_8$  consists of 8 rotations and 8 reflections.

Since, order of each reflection is 2.

$\therefore$  Number of elements of order 2 = 8 + 1 = 9.

**1.12 How many conjugacy classes does the permutation group  $S_5$  of permutations 5 numbers have? Write down one element in each class (preferably in terms of cycles).**

(2012 : 15 Marks)

**Solution:**

We know that two permutations are said to be conjugate if they have same cyclic decomposition.

There are 7 cyclic decompositions which are :

1.  $1 + 1 + 1 + 1 + 1$

[1 (5 times)]

2.  $1 + 1 + 1 + 2$

[2 (1 time), 1 (3 times)]

3.  $1 + 1 + 3$

[1 (2 times), 3 (1 time)]

4.  $1 + 4$

[1 (1 time), 4 (1 time)]

5.  $5$

[5 (1 time)]

6.  $1 + 2 + 2$

[1 (1 time), 2 (2 times)]

7.  $2 + 3$

[2 (1 time), 3 (1 time)]

- $\therefore$  Number of conjugacy classes = 7.  
 An element of 1st class = (1)  
 An element of 2nd class = (1 2)  
 An element of 3rd class = (1 2 3)  
 An element of 4th class = (1 2 3 4)  
 An element of 5th class = (1 2 3 4 5)  
 An element of 6th class = (1 2)(3 4)  
 An element of 7th class = (1 2 3)(4 5)

1.13 Give an example of an infinite group in which every element has finite order.

(2013 : 10 Marks)

Solution:

Consider  $\left\langle \frac{Q}{Z}, t \right\rangle$ , i.e., the quotient group of rationals with respect to integers under addition.

Any element of  $\frac{Q}{Z}$  is of form  $\frac{p}{q} + Z$  where  $\frac{p}{q} \in Q$  is in lowest form.

$$q\left(\frac{p}{q} + Z\right) = \underbrace{\frac{p}{q} + \dots + \frac{p}{q}}_{q \text{ times}} + Z = p + Z = Z$$

$$\therefore O\left(\frac{p}{q} + Z\right) \leq q$$

So, order of all elements are finite. But there are infinitely elements in  $\frac{Q}{Z}$  as there are infinite rationals in  $[0, 1)$  and none of them are equal in  $\frac{Q}{Z}$ .

$$\begin{aligned}
 \text{And } f[(a+ib)(c+id)] &= f[ac-bd+i(ad+bc)] \\
 &= \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
 &= f(a+ib) \cdot f(c+id)
 \end{aligned}$$

$f$  is one-one

$$a+ib = c+id \Rightarrow a=c \text{ and } b=d \Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} c & -d \\ d & a \end{bmatrix}$$

$\Rightarrow$

$$f(a+ib) = f(c+id)$$

$f$  is onto

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = f(a+ib)$$

$\therefore f$  is a one-one onto linear map and so an isomorphism.

1.14 What are the orders of the following permutations in  $S_{10}$ .

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix} \text{ and } (1 \ 2 \ 3 \ 4 \ 5) (6 \ 7)$$

(2013 : 10 Marks)

**Solution:**

**Approach :** This and the next part uses the fact that order of any permutation written as disjoint cycles in LCM of the order of the cycle. And order of any cycle is its length.

Writing the permutation as product of disjoint cycles.

$$(2\ 8)(3\ 7\ 4)(5\ 10\ 9\ 6)$$

$$\text{Order} = \text{L.C.M.}(2, 3, 4) = 12$$

$$(1\ 2\ 3\ 4\ 5)(6\ 7)$$

$$\text{Order} = \text{L.C.M.}(5, 2) = 10$$

**1.15** What is the maximal possible order of an element in  $S_{10}$ ? Why? Give an example of that element? How many elements will be there in  $S_{10}$  of that order?

(2013 : 13 Marks)

**Solution:**

Any permutation in  $S_{10}$  can be written as product of disjoint cycles. Let  $n_1, n_2, \dots, n_r$  be size of these cycles. Then they are also the order of respective cycles.

$$n_1 + n_2 + \dots + n_r = 10$$

and

$$\text{Order} = \text{L.C.M.}(n_1, n_2, \dots, n_r)$$

The order will be maximised if  $n_i$ 's are relative primes.

The primes less than 10 are 2, 3, 5, 7.

Taking  $n_1 = 3, n_2 = 7$ , Order = 21;  $n_1 = 2, n_2 = 3, n_3 = 5$ , Order = 30.

$\therefore$  The highest order is 30 for cycles of length 2, 3 and 5.

An example :  $(1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$

**Number of Such Elements :**

${}^{10}C_2$  ways of picking 2 elements for 1st cycle and they can be written  $(2-1)! = 1$  way.

${}^8C_3$  ways of picking 3 elements from rest 8 elements and can be written in  $(3-1)! = 2$  ways.

Rest 5 elements can be written in  $(5-1)! = 24$  ways.

$$\begin{aligned} \text{Total number of elements} &= {}^{10}C_2 \times 1 + {}^8C_3 \times 2 + 1 \times 24 \\ &= \frac{10 \times 9}{2} + \frac{8 \times 7 \times 6}{3 \times 2} \times 2 + 24 \\ &= 45 + 56 + 24 = 135 \text{ elements.} \end{aligned}$$

**1.16** Let  $G$  be the set of all real  $2 \times 2$  matrices  $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ , where  $xz \neq 0$ . Show that  $G$  group under matrix multiplication. Let  $N$  denote the subset  $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R} \right\}$ . Is  $N$  a normal subgroup of  $G$ ? Justify your answer.

(2014 : 10 Marks)

**Solution:**

Let  $(G, *)$  be an algebraic structure. Where  $|*$  implies multiplication between its element in ' $G$ ' is the set of

all  $2 \times 2$  matrices of type  $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}; xz \neq 0, x, y, z \in R$

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}; xz \neq 0, x, y, z \in R \right\}$$

(i) Closure Property:

$$\forall \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G$$

$$x, y, z, a, b \in 1R$$

...(i)

$$xz \neq a, ac \neq 0$$

...(ii)

$$\Rightarrow \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ax & xb + yc \\ 0 & zc \end{bmatrix} = A$$

$$ax \in 1R$$

$$(ax).(cz) = (ac)(xz) \neq 0 \text{ from (ii)}$$

$$xb + yc \in 1R$$

$$zc \in 1R \text{ from (i)}$$

$\therefore$

$$A \in G \Rightarrow (G, *) \text{ is closed}$$

(ii) Associative Property :

$$\forall \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} \in G$$

$$x, y, z, a, b, c, p, r \in 1R$$

$$xz \neq 0, ac \neq 0, Pr \neq 0$$

$$\Rightarrow \left( \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} = \begin{bmatrix} ax & bx + yc \\ 0 & zc \end{bmatrix} * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix}$$

$$= \begin{bmatrix} apx & aqx + bxr + ycr \\ 0 & zcr \end{bmatrix}$$

...(iii)

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} \right) = \begin{bmatrix} apx & aqx + bxr + ycr \\ 0 & zcr \end{bmatrix}$$

...(iv)

Here, (iii) = (iv)

$$\text{i.e.} \quad \left( \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} \right)$$

$\therefore (G, x)$  satisfy associative property.

(iii) Existence of Left Identity :

$$\text{Let} \quad \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in G, \text{ where } xz \neq 0, x, y, z \in 1R$$

...(v)

$$\text{Let} \quad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G, \text{ where } ac \neq 0, a, b, c \in 1R$$

...(vi)

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$\begin{bmatrix} ax & ay + bz \\ 0 & cz \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

equating its element,

$$cz = z$$

$\Rightarrow$

$$z(c - 1) = 0$$

From (vi),

$$ac \neq 0$$

$\Rightarrow$

$$a \neq 0, c \neq 0$$

From (v)

$$xz \neq 0$$

$\Rightarrow$

$$x \neq 0, z \neq 0$$

$\therefore$

$$c - 1 = 0$$

$\Rightarrow$

$$c = 1$$

$$ax = x$$

$\Rightarrow$

$$x(1 - a) = 0$$

From (v)  $x \neq 0$

$$1 - a = 0$$

$$a = 1$$

$$ay + bz = y$$

$$y + bz = y$$

$\Rightarrow$

$$bz = 0$$

From (v)

$$z \neq 0$$

$\therefore$

$$b = 0$$

$\therefore$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{left identity.}$$

(iv) Existence of left inverse:

Let

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \in G$$

Where,

$$xz \neq 0, pr \neq 0$$

$$x, y, z, p, q, r \in 1R$$

$$\neq 0$$

$$\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} * \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} px & py + qz \\ 0 & rz \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

equating the elements,

$$px = 1$$

$$p = \frac{1}{x}$$

$$rz = 1$$

$$r = \frac{1}{z}$$

$$py + qz = 0$$

$$\frac{y}{x} + az = 0$$

$$q = -\left(\frac{y}{xz}\right)$$

$\therefore$

$$\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & -\frac{y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix} \text{ left inverse of } \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

...(vii)

$\therefore (G, *)$  is a group,

$$N = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}; a \in 1R \right\} \subset G$$

$$e = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in N; N \neq \phi$$

Let,

$$A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in N; x \in 1R$$

$$A^{-1} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$$

From (vii)

(i)  $A^{-1} \in N$

(ii) Let,

$$B = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \in N; y \in 1R$$

$$A * B = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y+x \\ 0 & 1 \end{bmatrix} \in N$$